Fibonacci Series

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Abstract

This paper provides an overview of the Fibonacci and Lucas series in math, covering their definitions, patterns, and features. We explore their recursive nature, formulas, and forms. We study their math traits like divisibility and congruences. This paper aims to deepen understanding in Fibonacci Series.

1 Introduction

Recurrence Relation

1. A recurrence relation is an equation which represents a sequence based on some rule which allows us to find the whole sequence.
2. It's main property is that we can find the n^{th} term if we know the $n-1^{th}$ term (one - fold) or on the basis of $(n-1)^{th}$ and $(n-2)^{th}$ term (two-fold).

Fibonacci Series -

1. The fibonacci series is a simple series which is found by applying a recursive formula $F_n = F_{n-1} + F_{n-2}$.
2. However, as with all recursive formulae we need a base case. As this has a two-fold recursion (F_n depends on F_{n-1} and F_{n-2}) there will be two values comprising the base, $F_1 = 1, F_2 = 1$.

Lucas Series -

1. The lucas series too is a recursive series of the form $L_n = L_{n-1} + L_{n-2}$.
2. However, the base numbers must be different as else the two series would be the same as the recursive formula is constant.
3. Thus the base numbers are $L_1 = 1, L_2 = 3$.

2 General form of Fibonacci series

1. We know, $F_n = F_{n-1} + F_{n-2} \Rightarrow F_n - F_{n-1} - F_{n-2} = 0$
2. We consider the quadratic equation $x^2 - x - 1 = 0$
3. This can also be written as $x^2 = x + 1$
4. We write this way for all the terms upto x^n
   - $x = x$
   - $x^2 = x + 1$
   - $x^3 = x \cdot x^2 = x \cdot (x+1) = x^2 + x = x + 1 + x = 2x + 1$
   - $x^4 = x \cdot x^3 = x \cdot (2x+1) = 2x^2 + 2x = 2(x+1) + x = 3x + 2$
   - $x^5 = 5x + 3$
   - $x^6 = 8x + 5$
   - $x^n = F_n \cdot x + F_{n-1}$
5. Consider the roots of the original $x^2 - x - 1 = 0$ which gives us $x = \left\{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right\}$.
6. Let $\Phi = \frac{1+\sqrt{5}}{2}$ and $\Psi = \frac{1-\sqrt{5}}{2}$. Since these both Φ and Ψ are roots of the quadratic they satisfy $x^n = F_n \cdot x + F_{n-1}$
7. Using this, we get
$$\Phi^n = F_n \cdot \Phi + F_{n-1}$$
$$\Psi^n = F_n \Psi + F_{n-1}$$
8. Subtracting the second equation from the first we get that
$$\Phi^n - \Psi^n = F_n \cdot (\Phi - \Psi)$$
9. $$F_n = \frac{\Phi^n - \Psi^n}{\sqrt{5}}$$
10. Therefore,
$$F_n = \frac{1}{\sqrt{5}} \cdot \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

3 Sum of n terms

Theorem 1 *The sum of the fibonacci series of n terms*

$$\forall n \in \mathbb{N} \rightarrow \sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Proof

1. Let

$$\Phi = \frac{1+\sqrt{5}}{2}, \Psi = \frac{1-\sqrt{5}}{2}$$

2.

$$\sum_{i=1}^{n} F_i = \frac{1}{\sqrt{5}} \sum_{i=1}^{n} (\Phi)^n - \frac{1}{\sqrt{5}} \sum_{i=1}^{n} (\Psi)^n$$

3.

$$\sum_{i=1}^{n} F_i = \frac{1}{\sqrt{5}} \left[(\Phi) \cdot \frac{(\Phi)^n - 1}{(\Phi) - 1} - (\Psi) \cdot \frac{(\Psi)^n - 1}{(\Psi) - 1} \right]$$

4.

$$\sum_{i=1}^{n} F_i = \frac{1}{\sqrt{5}} \left[(\Phi) \cdot \frac{(\Phi)^n - 1}{(\Psi)} - (\Psi) \cdot \frac{(\Psi)^n - 1}{(-\Phi)} \right]$$

5.

$$\sum_{i=1}^{n} F_i = \frac{1}{\sqrt{5}} \left[(\Phi)^{n+2} - (\Psi)^{n+2} - (\Phi + 1) + \Psi + 1 \right]$$

6.

$$\sum_{i=1}^{n} F_i = \frac{1}{\sqrt{5}} \left[(\Phi)^{n+2} - (\Psi)^{n+2} \right] - 1$$

7.

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

Proof by induction

1. **Base Case** - We have to show that

$$n = 1 \Rightarrow \sum_{i=0}^{1} F_i = F_3 - 1$$

We know $F_1 = 1$ and $F_3 - 1 = 2 - 1 = 1$
Hence the base case is true.

2. **Induction Hypothesis** - We assume that for $k \in \mathbb{N}, k > 1$

$$\sum_{i=0}^{k} (F_i)^2 = F_{k+2} - 1$$

3. **Induction Step** -

- We have to prove for $k + 1$ so,

$$\sum_{i=1}^{k+1} F_i = \sum_{i=1}^{k} F_i + F_{k+1}$$

- By Induction Hypothesis,

$$\sum_{i=1}^{k+1} F_i = F_{k+2} - 1 + F_{k+1}$$

- By definition of F_{k+3},

$$\sum_{i=1}^{k+1} F_i = F_{k+3} - 1$$

- This proves the claim for F_{k+1} which completes the proof

4 Squared terms

Theorem 2 *The difference of squares of terms also is in the series*

$$\forall n \in \mathbb{N}n\{1,2\} \rightarrow (F_n)^2 - (F_{n-2})^2 = F_{2(n-1)}$$

Proof

1. $$(F_n)^2-(F_{n-2})^2 = \frac{1}{5}\cdot\left[(\Phi)^{2n} + (\Psi)^{2n} - 2\left(\Phi \cdot \Psi\right)^{n}\right]-\frac{1}{5}\cdot\left[(\Phi)^{2n-4} + (\Psi)^{2n-4} - 2\left(\Phi \cdot \Psi\right)^{n-2}\right]$$

2. $$(F_n)^2 - (F_{n-2})^2 = \frac{1}{5} \cdot \left[\left((\Phi)^4 - 1\right)(\Phi)^{2n-4} + \left((\Psi)^4 - 1\right)(\Psi)^{2n-4}\right]$$

3. $$(F_n)^2 - (F_{n-2})^2 = \frac{1}{5} \cdot \left[(3\Phi + 1)(\Phi)^{2n-4} + (3\Psi + 1)(\Psi)^{2n-4}\right]$$

4. $$(F_n)^2 - (F_{n-2})^2 = \frac{1}{\sqrt{5}} \cdot \left[(\Phi + 1)(\Phi)^{2n-4} + (\Phi - 2)(\Psi)^{2n-4}\right]$$

5. $$(F_n)^2 - (F_{n-2})^2 = \frac{1}{\sqrt{5}} \cdot \left[(\Phi)^2 (\Phi)^{2n-4} - (\Psi)^2 (\Psi)^{2n-4}\right]$$

6. $$(F_n)^2 - (F_{n-2})^2 = \frac{1}{\sqrt{5}} \cdot \left[(\Phi)^{2n-2} - (\Psi)^{2n-2}\right]$$

7. $$(F_n)^2 - (F_{n-2})^2 = F_{2(n-1)}$$

5 Product of alternative terms

Theorem 3 *The product of alternative terms ± 1 is also in the series*

$$F_{n-1} \cdot F_{n+1} = (F_n)^2 + (-1)^n$$

Proof

1.
$$(F_{n-1})(F_{n+1}) = \frac{1}{5}\left[\left[(\Phi)^{n+1} - (\Psi)^{n+1}\right]\left[(\Phi)^{n-1} - (\Psi)^{n+1}\right]\right]$$

2.
$$(F_{n-1})(F_{n+1}) = \frac{1}{5}\left[(\Phi)^{2n} + (\Psi)^{2n} - (-1)^{n-1} \cdot (\Phi + 1 + \Psi + 1)\right]$$

3.
$$(F_{n-1})(F_{n+1}) = \frac{1}{5}\left[(\Phi)^{2n} + (\Psi)^{2n} - 3(-1)^{n-1}\right]$$

4.
$$(F_{n-1})(F_{n+1}) = \frac{1}{5}\left[(\Phi)^{2n} + (\Psi)^{2n} + 2(-1)^{n-1}\right] - (-1)^{n-1}$$

5.
$$(F_n)^2 = \frac{1}{5} \cdot \left[(\Phi)^{2n} + (\Psi)^{2n} - 2 \cdot (\Phi \cdot \Psi)^n\right]$$

6.
$$(F_n)^2 = \frac{1}{5} \cdot \left[(\Phi)^{2n} + (\Psi)^{2n} - 2 \cdot (-1)^n\right]$$

7.
$$(F_n)^2 = \frac{1}{5} \cdot \left[(\Phi)^{2n} + (\Psi)^{2n} + 2 \cdot (-1)^{n-1}\right]$$

8.
$$F_{n+1} \cdot F_{n-1} + (-1)^{n-1} = (F_n)^2 \Rightarrow F_{n+1} \cdot F_{n-1} = (F_n)^2 + (-1)^n$$

6 Divisibility by a number n

Theorem 4

$$n \mid F_k, F_{k+1} \equiv 1 \ (mod\ n) \Rightarrow n \mid \sum_{i=1}^{k} F_i$$

Proof

1. $$\exists k_1 \text{ s.t. } F_k = n \cdot k_1$$

2. $$\exists k_2 \text{ s.t. } F_{k+1} = n \cdot k_2 + 1$$

3. $$F_{k+2} = F_{k+1} + F_k$$

4. $$F_{k+2} = n(k_1 + k_2) + 1$$

5. $$F_{k+2} - 1 = n(k_1 + k_2)$$

6. $$\sum_{i=1}^{k} F_i = n(k_1 + k_2)$$

7. $$n \mid \sum_{i=1}^{k} F_k$$

7 Sum of squared terms

Theorem 5 *The sum of squares of terms also is product of two numbers of the series*

$$\forall n \in \mathbb{N} \rightarrow \sum_{i=0}^{n} (F_i)^2 = F_n \cdot F_{n+1}$$

Proof by induction

1. **Base Case** - $n = 1 \Rightarrow (F_1)^2 = F_1 \cdot F_2$ as $1^2 = 1 \cdot 1$
2. **Induction Hypothesis** - We assume that for $k \in \mathbb{N}, k > 1$

$$\sum_{i=0}^{k} (F_i)^2 = F_k \cdot F_{k+1}$$

3. **Induction Step** - By Induction Hypothesis,
   - We have assume the property to be true for k. So,

$$\sum_{i=1}^{k} (F_i)^2 = F_k \cdot F_{k+1}$$

   - Since we have to prove the claim true for F_{k+1}, we add its square to both sides. So,

$$\sum_{i=1}^{k} (F_i)^2 + (F_{k+1})^2 = (F_k \cdot F_{k+1}) + F_{k+1}^2$$

   - Factoring out F_{k+1} we get,

$$\sum_{i=1}^{k+1} (F_i)^2 = (F_{k+1} \cdot (F_k + F_{k+1}))$$

   - By definition $F_k + F_{k+1} = F_{k+2}$. So,

$$\sum_{i=1}^{k+1} (F_i)^2 = F_{k+1} \cdot F_{k+2}$$

   - This proves the claim for F_{k+1} which completes the proof

Alternate Proof

1.
$$\sum_{i=0}^{n}(F_i)^2 = \frac{1}{5}\left[\sum_{i=1}^{n}\Phi^{2n} + \sum_{i=1}^{n}\Psi^{2n} + 2(-1)^{n+1}\right]$$

2.
$$\sum_{i=0}^{n}(F_i)^2 = \frac{1}{5}\left[\Phi^2 \cdot \left(\frac{\Phi^{2n}-1}{\Phi^2-1}\right) + \Psi^2 \cdot \left(\frac{\Psi^{2n}-1}{\Psi^2-1}\right)\right] + 2(-1)^{n+1}$$

3.
$$\sum_{i=0}^{n}(F_i)^2 = \frac{1}{5}\left[\Phi^{2n+1} + \Psi^{2n+1} - 1 + 2 \cdot \sum_{i=1}^{n}(-1)^{i+1}\right]$$

4. If n is even
$$\sum_{i=0}^{n}(F_i)^2 = \frac{1}{5} \cdot \left[\Phi^{2n+1} + \Psi^{2n+1} - 1\right]$$

5. If n is odd
$$\sum_{i=0}^{n}(F_i)^2 = \frac{1}{5} \cdot \left[\Phi^{2n+1} + \Psi^{2n+1} + 1\right]$$

6.
$$\sum_{i=0}^{n}(F_i)^2 = \frac{1}{5} \cdot \left[\Phi^{2n+1} + \Psi^{2n+1} + (-1)^{n+1}\right]$$

7.
$$F_n \cdot F_{n+1} = \frac{1}{5} \cdot [\Phi^n - \Psi^n] \cdot \left[\Phi^{n+1} - \Psi^{n+1}\right] = \frac{1}{5} \cdot \left[\Phi^{2n+1} + \Psi^{2n-1} + (-1)^{n+1} \cdot (\Phi + \Psi)\right]$$

8.
$$F_n \cdot F_{n+1} = \frac{1}{5} \cdot \left[\Phi^{2n+1} + \Psi^{2n-1} + (-1)^{n+1}\right]$$

9.
$$\sum_{i=0}^{n}(F_i)^2 = F_n \cdot F_{n+1}$$

8 Closure of powers of general term

Theorem 6

$$\forall m \in \mathbb{N} \rightarrow \Phi^m + \Psi^m \in \mathbb{Z}$$

Proof

1. **Base Case 1** - $m = 1 \Rightarrow \Phi + \Psi = 1$ and $1 \in \mathbb{Z}$
2. **Base Case 2** - $m = 2 \Rightarrow \Phi^2 + \Psi^2 = 3$ and $3 \in \mathbb{Z}$
3. **Induction Hypothesis** - We assume that

$$(\Phi)^{m-1} + (\Psi)^{m-1} \in \mathbb{Z}$$

$$(\Phi)^m + (\Psi)^m \in \mathbb{Z}$$

4. **Induction Step** -
   - We wish to prove for $m + 1$. So,

$$\Phi^{m+1} + \Phi^{m+1} = (\Phi^m + \Phi^m) \cdot (\Phi + \Psi) - \Phi\Psi \left(\Phi^{m-1} + \Phi^{m-1}\right)$$

   - We know

$$\Phi + \Psi = 1$$

$$\Phi\Psi = -1$$

   - Let ζ represent an integer. So,

$$\Phi^{m+1} + \Phi^{m+1} = \zeta(1) - (-1)\zeta$$

   - Therefore,

$$\Phi^{m+1} + \Phi^{m+1} = 2\zeta \Rightarrow \Phi^{m+1} + \Phi^{m+1} \in \mathbb{Z}$$

   - This proves the claim for $m + 1$ which completes the proof

9 Divisibility between terms

Theorem 7

$$\forall a \in \mathbb{Z}^+ \to F_k \mid F_{a \cdot k}$$

Proof

1. **Base Case 1** - $m = 1 \Rightarrow F_a \mid F_a$ which is true as $F_i \in \mathbb{Z}$ and $\forall \alpha \in \mathbb{Z} \to \alpha \mid \alpha$
2. **Base Case 2** - $m = 2 \Rightarrow F_a \mid F_{2a}$ which must be true.
   - By definition of F_i,
   $$F_{2a} = \frac{1}{\sqrt{5}} \cdot \left(\Phi^{2a} - \Psi^{2a}\right)$$
   - This is also,
   $$F_{2a} = \frac{1}{\sqrt{5}} \cdot (\Phi^a - \Psi^a) \cdot (\Phi^a + \Psi^a)$$
   - By the general form,
   $$F_{2a} = F_a \cdot (\Phi^a + \Psi^a)$$
   - By definition of multiple,
   $$F_a \mid F_{2a}$$
3. **Induction Hypothesis** - We assume that
   $$F_a \mid F_{a(m-1)}$$
   $$F_a \mid F_{a(m)}$$
4. **Induction Step** -
   - We wish to prove for $m + 1$. So,
   $$F_{a(m+1)} = \frac{1}{\sqrt{5}} \cdot \left(\Phi^{am+a} - \Psi^{am+a}\right)$$
   - This is also,
   $$F_{a(m+1)} = \frac{1}{\sqrt{5}} \cdot (\Phi^{am} - \Psi^{am}) \cdot \left(\Phi^a + \Psi^a\right) - (\Phi\Psi)^a \cdot \left(\Phi^{a(m-1)} - \Psi^{a(m-1)}\right)\Big)$$
   - So,
   $$F_{a(m+1)} = F_{am}(\Phi^a + \Psi^a) - (-1)^a \cdot F_{a(m-1)}$$

- By Theorem 6,
$$\Phi^a + \Psi^a \in \mathbb{Z}$$
- By Induction Hypothesis,
$$F_a \mid F_{a(m-1)}$$
$$F_a \mid F_{a(m)}$$
- Therefore,
$$F_a \mid F_{a(m+1)}$$
- This proves the claim for $m + 1$ which completes the proof

10 Congruence modulo a prime p

Theorem 8

$$\forall \ primes \ p \setminus 2,5 \rightarrow F_p \equiv \pm 1 \ mod(p)$$

Proof

1. $$F_p = \frac{1}{\sqrt{5}} \cdot (\Phi^p - \Psi^p)$$

2. $$F_p = \frac{1}{\sqrt{5}} \cdot \left(\frac{2(\sqrt{5}) \cdot \binom{p}{1} + 2(\sqrt{5})^3 \cdot \binom{p}{3} + \cdots 2(\sqrt{5})^p \cdot \binom{p}{p}}{2^p} \right)$$

3. $$F_p = \frac{1}{2^{p-1}} \cdot \left(\binom{p}{1} + 5 \cdot \binom{p}{3} + 5^2 \cdot \binom{p}{5} + \cdots 5^{\frac{p-1}{2}} \cdot \binom{p}{p} \right)$$

4. So, in Z_p as GCD(2, p) = 1 by Fermat's Little Theorem, $2^{p-1} \equiv 1$

5. $$F_p = \left(\binom{p}{1} + 5 \cdot \binom{p}{3} + 5^2 \cdot \binom{p}{5} + \cdots 5^{\frac{p-1}{2}} \cdot \binom{p}{p} \right)$$

6. We know that

$$p \mid \binom{p}{k} \Rightarrow \left(\binom{p}{1} + 5 \cdot \binom{p}{3} + 5^2 \cdot \binom{p}{5} + \cdots 5^{\frac{p-1}{2}} \cdot \binom{p}{p} \right) = 5^{\frac{p-1}{2}}$$

7. Also, If 5 is a square in U_p if $5^{\frac{p-1}{2}} = 1$ and 5 is a non square in U_p if $5^{\frac{p-1}{2}} = -1$

8. Therefore,

$$F_p \equiv \pm 1 \text{ mod(p)}$$

11 Relation between Fibonacci and Lucas Series

Theorem 9

$$L_n = F_{n-1} + F_{n+1}$$

Proof

1. **Base Case 1** - $n = 2 \Rightarrow L_n = L_2 = 3$ Also, $F_1 + F_3 = 1 + 2 = 3$
2. **Base Case 2** - $n = 3 \Rightarrow L_n = L_3 = 4$ Also, $F_2 + F_4 = 1 + 3 = 4$
3. **Induction Hypothesis** - We assume that

$$L_k = F_{k-1} + F_{k+1}$$

$$L_{k-1} = F_{k-2} + F_k$$

4. **Induction Step** -
   - By Induction Hypothesis we know

     $$L_k = F_{k-1}+F_{k-1}, L_{k-1} = F_{k-2}+F_k \Rightarrow L_k+L_{k-1} = F_{k+1}+F_k+F_{k-1}+F_{k-2}$$

   - By definition,

     $$L_k + L_{k-1} = F_{k+2} + F_k$$

   - This proves the claim for $n + 1$ which completes the proof

12 Order or Period of cyclicity

Order - The order of the Fibonacci Series O_n is defined as the number of terms after which the series repeats. It is also called period of cyclicity.

1. Clearly, it will be noted that as $F_n = F_{n-1} + F_{n-2}$ and $F_1 + F_2 = F_3 > 0$ the series will never repeat as it is strictly increasing.

2. Therefore, we only define order in regard to modulo a number n which is mentioned as a parameter for the order function O

3. An example for this would be the fibonacci series modulo 5 which is

$$\{1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0\}$$

   after which the series repeats.

4. Therefore, $O_5 = 20$

```
# Order Calculator
fibonacci = [1, 1]
num_rep = 0
counter = 1
modulus = 5
while num_rep != 2:
    if fibonacci[counter] == fibonacci[counter-1] == 1:
        num_rep += 1
    next_num = (fibonacci[counter] + fibonacci[counter-1]) % modulus
    fibonacci.append(next_num)
    counter += 1
print("Order is " + str(len(fibonacci)-3))
```

Theorem 10 *Let $n = p_1 \cdot p_2 \cdots p_n$ where p_i for $1 \leq i \leq n$ are primes. For each p_i if O_i is the order then the order of n i.e., $O_n = lcm(O_1, O_2, \cdots O_k)$*

Proof

1. Since $n = p_1 \cdot p_2 \cdots p_n$ where p_i for $1 \leq i \leq n$ are primes.
2. $F_{O_n} \equiv 0 \pmod{n}$ and $F_{O_{n+1}} \equiv 1 \pmod{n} \Rightarrow F_{O_n} \equiv 0 \pmod{P_i}$ and $F_{O_{n+1}} \equiv 1 \pmod{P_i}$
3. Hence, O_n must be a multiple of O_i.
4. Hence, $O_i \mid O_n, \forall i \in \{1, 2, \cdots k\}$
5. Hence, the order (minimum of such values) will be the $lcm(O_1, O_2 \cdots O_k)$

Theorem 11

$$O_{ab} = lcm(O_a, O_b) \text{ if } gcd(a, b) = 1$$

Proof

1. $$F_{O_{ab}} \equiv 0 \pmod{ab}$$
   $$F_{O_{ab+1}} \equiv 1 \pmod{ab}$$
2. $$F_{O_a} \equiv 0 \pmod{a}$$
   $$F_{O_{a+1}} \equiv 1 \pmod{a}$$
3. $$O_a, O_b \mid O_{ab}$$
4. Hence order will be
   $$\text{lcm}(a, b)$$

13 Conjectures

13.1 Order of numbers modulo composite numbers

1. We consider the first few composite numbers beginning from 4 till 12 and analyse the order of these along with their sum uptill cyclicity and that sum modulo the composite number.

Modulo (c)	Order	Sum	Sum (mod c)
4	6	8	0
6	24	66	0
8	12	32	0
9	24	108	0
10	60	280	0
12	24	108	0

Conjecture 1 *Forall composite numbers c,* $6 \mid O_c$

Conjecture 2 *Forall composite numbers c, if* $3 \mid c \Rightarrow O_c = 24$

Counterexamples - $c = 15, 21 \Rightarrow O_c = 40, 16$

Conjecture 3

$$S_c \equiv 0 (mod\ c)$$

In other words the third conjecture is saying that the last column of the table is always going to be 0.

13.2 Conversion to binary

1. Consider the terms of the Fibonacci Series modulo 2

$$F : 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 \cdots$$

2. Convert each of these terms into binary and observe the number of 1's which occur in it.

   - $F_{b_1} = 1, F_{b_2} = 1$
   - $F_{b_3} = 10, F_{b_4} = 11$
   - $F_{b_5} = 101, F_{b_6} = 1000$
   - $F_{b_7} = 1101, F_{b_8} = 10101$
   - $F_{b_9} = 100010, F_{b_{10}} = 110111$
   - $F_{b_{11}} = 1011001, F_{b_{12}} = 10010000$
   - $F_{b_{13}} = 11101001, F_{b_{14}} = 101111001$

3. We consider the following table

F_b	**No. Ones**
1	1
2	1
3	1
4	1
5	2
6	1
7	3
8	3
9	2
10	5
11	4
12	2
13	5
14	6

Conjecture 4 *The number of ones in F_b is completely random but always is a natural number.*